Field Theory Revisited

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Abstract: Following P. A. M. Dirac's critique in "Lectures on Quantum Field Theory" of the usual formalism, I will discuss the role of the time parameter to solve R. Haag's no-go theorem on the non-equivalence of the conventional Schrödinger picture and the Heisenberg picture. This is possible by first defining in a correct way the concept of the vacuum state at a given time in relativity. I will also discuss some consequences such as the spectral condition.

We will take as our basic considerations P. A. M. Dirac's beautiful lectures on Quantum Field Theory delivered at the Yeshiva University during the academic year 1963 – 1964 and published by Academic Press in 1966[(1)]. We will also suppose that the reader knows the abstract Fock space construction as developed along the line introduced by D. Kastler[(2)]. In his lectures cited above Dirac writes [p.6]:

> *We have the kets at one particular time and we may picture them as corresponding to physical states at that time, so we retain the concept of a physical state at a certain time. We do not have the concept of a physical state throughout all time. The latter concept, which lies at the basis of the Schrödinger picture, will not occur in the present formulation.*

This expresses exactly or point of view, where states are states at the actual time, and the time is a c-number taking a well defined value in any possible state. This is in marked contrast to q-numbers, which have in general no values in any state. In spite of appearances, the noncommutativity of the q-numbers is not the very essence of such physical objects. In fact this depends on the mathematical formalism that you choose and its corresponding interpretation. This can be seen very well in the Wigner representation of p and q. The two objects commute with each other but the usual rules of quantum mechanics are here completely modified and nevertheless in any given physical state p and q have no values.

The concept of a physical state throughout all time, a relativistic concept, is not a state at all, it is a trajectory of states labeled by the time, and a solution of a Schrödinger equation[(3)]. But as Dirac demonstrates, such solutions do not exist for the physically justified Hamiltonian for which the interaction is too violent at high frequencies.

To exhibit the difficulties and explain Dirac's preference for the Heisenberg picture, let us take the very simple example of a fermionic field (non-self-interacting and non-relativistic). Consider a one-particle problem is non-relativistic quantum mechanics. As we know, such a system is described by a family of Hilbert spaces labeled by some c-number α. In the most simple case α is just the time, which is as Dirac insists a c-number[(4)]. In the general formalism[(5)], each physical observable is described by a family of self adjoint operators or better by a the corresponding spectral families. According to G. W. Mackey, the observables $\underline{p}$, $\underline{q}$, $\underline{t}$ are solutions of the imprimitivity systems based on the kinematic group of rotations and translations of $\vec{p}$, $\vec{q}$ and t and we have the following representation:

I

$$\underline{q}: \quad \{q_t = x\}$$
$$\underline{p}: \quad \{ p_t = -i\hbar\partial_x \}$$
$$\underline{t}: \quad \{t_t = tI\}$$

This representation is called the Schrödinger representation since the time t does not appear explicitly in the operators describing $\underline{p}$ and $\underline{q}$ and more precisely it is the representation in the $\vec{q}$ variable (with diagonal $\underline{q}$).

As we have said, the state is the state at a given time t and it is described by a ray $\varphi_t(x)$ in the Hilbert space H_t, the isomorphism which corresponds to the translation of time in the imprimitivity relations is a passive translation which allows the comparison between $\varphi_t(x)$ in H_t and $\varphi_{t+\tau}(x)$ in $H_{t+\tau}$, it is not the evolution which is an active translation from t to $t+\tau$. But such passive translations give meaning to the Schrödinger equation

$$\mathrm{i}\mathrm{d}_t\varphi_t(x) = H_t\varphi_t(x) \tag{1}$$

where H_t is the Schrödinger operator which is self adjoint when the evolution is induced by a unitary transformation. In this particular case we can change from the Schrödinger representation to the corresponding Heisenberg representation. For example:

The Heisenberg representation for the free particle

II

$$\underline{q}: \quad \{ q_t = x + \tfrac{1}{m}(-i\hbar\partial_x)t \}$$
$$\underline{p}: \quad \{ p_t = -i\hbar\partial_x\}$$
$$\underline{t}: \quad \{t_t = tI\}$$

The Heisenberg representation for the harmonic oscillator

III

$$\underline{q}: \quad \{q_t = \cos\omega t\, x + \tfrac{1}{m\omega}\sin\omega t(-i\hbar\partial_x) \}$$
$$\underline{p}: \quad \{ p_t = \cos\omega t(-i\hbar\partial_x) - m\omega\sin\omega t\, x \}$$
$$\underline{t}: \quad \{t_t = t\, I\}$$

We go from the Schrödinger to the Heisenberg representation by a unitary transformation labeled by t but acting in each Hilbert space separately:

$$(\mathrm{I}) \xrightarrow{e^{\frac{i}{\hbar}(\frac{1}{2m}p^2)t}} (\mathrm{II}) \qquad\qquad (\mathrm{I}) \xrightarrow{e^{\frac{i}{\hbar}(\frac{1}{2m}p^2 + \frac{m\omega}{2}q^2)t}} (\mathrm{III})$$

To be able to apply the resources of functional analysis we have to restrict $\varphi_t(x)$ for each t to be in $\mathcal{S}(R^3)$, the subspace of smooth functions of rapid decrease. But this is not enough, we have to consider also a bigger space $\mathcal{H} = \int_{\oplus} H_t dt$ and restrict ourselves to $\varphi(t,x) \in \mathcal{S}(R^4)$. In this context, the solutions of the Schrödinger equation (1) are in fact in $\mathcal{S}'(R^4)$ and the operator $K = i\partial_t - H_t$ acting on such $\mathcal{H}$ has continuous spectrum from $-\infty$ to ∞. The Schrödinger solution must be interpreted as a generalised eigenvector for the eigenvalue 0:

$$K\varphi(t,x) = 0 \tag{2}$$

Consequently, in $\mathcal{H}$ the operator K is unitarily equivalent to the 'trivial one' $i\partial_t$. It is only in this bigger space $\mathcal{H}$ that we can give a meaning to relativistic covariance, but we have to interpret everything at a given time t_0 and as Dirac explains [p.6]:

> *For example, take the equation $\alpha(t_0)|A\!> = a|A\!>$ where a is a number. If we had that equation, we could say that $|A\!>$ represents the state at time t_0 for which the dynamical variable α at time t_0 certainly has the value a.*

Such an interpretation of eigenvalues and eigenvectors is exactly the one that we have always given.

Knowing the description of the one-particle states, we can define the N-particle states by the Fock construction, for each value of the c-number t we can build the Fock space $\mathcal{F}(H_t)$, the space $\oplus_n (H_t)^{\otimes n}$ after symmetrisation or antisymmetrisation, and the corresponding creation and annihilation operators $a^\dagger(\varphi_t)$ and $a(\varphi_t)$. By taking the direct integral you can then construct a bigger Fock space, once again the good space in which to implement the relativistic covariance. This gives the beginnings of a new field theory.

Let us conclude with some remarks on such a revisited field theory.

- In complete analogy with the notion introduced by Dirac [p.147], at each time t we can define the vacuum $|0_t\!>$ by the condition that $a(\varphi_t)|0_t\!> = 0$ for any $a(\varphi_t)$. Obviously such a family $|0_t\!>$ is not unique (any $e^{\subset \alpha(t)}\,|0_t\!>$ is another solution), it is even not normalisable being in fact in $\mathcal{S}'(R)$. Such a vacuum differs very drastically from the usual concept and here in may cases (in particular the examples given above) the Heisenberg and Schrödinger representations are unitarily equivalent.
- The usual spectral condition must be modified. Here the operator of the generator corresponding to the time-translation evolution is unbounded in both directions but degenerate starting from some lower bound in energy.
- The q-number parts of the field are not just q-number Schwartz distributions but q-number de Rham currents(6), which means, among other things, that that the test functions must be replaced by the one-particle state functions in $\mathcal{S}(R^4)$.
- The usual relativistic dynamical covariance of the Poincaré group defines isomorphisms of the Hilbertian structure of the global property lattice which, in general, are implemented by non-unitary and non-irreducible representations due to the fact that the Poincaré group acts also on the c-number part of the field(7).

REFERENCES

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(3) C. Piron "Mécanique quantique bases et applications" Presses polytechniques et universitaires romandes, Lausanne, 1998, ch.6

(4) See also W. Pauli "General Principles of Quantum Mechanics" Springer-Verlag, New York, 1980, p.63

(5) C. Piron "Mécanique quantique bases et applications" Presses polytechniques et universitaires romandes, Lausanne, 1998, ch.3

(6) G. de Rham "Variétés différentiables" Hermann, Paris, 1960, §§8 and 31– 32

(7) G. C. D'Emma "On quantization of the electromagnetic field" *Helvetica Physica Acta* **53** (1980) 535-551