Morphisms, Hemimorphisms and Baer *-Semigroups

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The relationship between CROCs (complete orthomodular lattices) and complete Baer *-semigroups is discussed using an explicit construction of the adjoint of a hemimorphism. Simple examples provide much insight into the structures involved.

1. Preliminaries

A CROC is nothing else than a complete orthomodular lattice (Piron 1976). We call it a CROC as it is canonically relatively orthocomplemented, which means that each segment $[0, a]$, that is the set of elements between 0 and a, is by itself a complete orthomodular lattice, where the orthocomplementation is defined by $x^r = x' \wedge a$. A hemimorphism from a CROC $\mathcal{A}$ to a CROC $\mathcal{B}$ is a map ϕ from $\mathcal{A}$ to $\mathcal{B}$ which maps 0 to 0 and preserves the supremum:

$$\phi(\vee_i a_i) = \vee_i \phi(a_i).$$

According to the usual definitions a hemimorphism which conserves the orthogonality relation is called a morphism.

A complete Baer *-semigroup is a set S equipped with (Foulis 1960, see also Pool 1968)

(i) an associative multiplication law with a (necessarily unique) 0 and I:

$$(fg)h = f(gh),$$
$$0f = f0 = 0 \qquad \forall f \in S,$$
$$If = fI = f \qquad \forall f \in S,$$

and

(ii) an involution $f \mapsto f^*$:

$$f^{**} = f,$$
$$(fg)^* = g^* f^*,$$

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in such a way that

(iii) each annihilator $\{f \mid fg = 0 \quad \forall g \in M \subset S\}$ is an ideal of the form Sp, where p is a projection, that is an element of S such that

$$p = p^* = p^2.$$

One can easily show that 0 and I are projections and that the annihilator of 0 is generated by I and that of I by 0.

The aim of the next two sections is to show that these two structures are intimately linked. Finally we consider an instructive example in section 4.

2. The complete Baer *-semigroup associated to a CROC

Let $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{A}$ be hemimorphisms. Then by definition ϕ and ψ form an adjoint pair if the following two conditions are satisfied:

$$\psi(\phi a)' < a' \qquad \forall a \in \mathcal{A},$$
$$\phi(\psi b)' < b' \qquad \forall b \in \mathcal{B}.$$

Surprisingly, given any hemimorphism ϕ there exists a hemimorphism ψ such that ϕ and ψ form an adjoint pair. More precisely we have

Lemma: Each hemimorphism $\phi : \mathcal{A} \to \mathcal{B}$ has a unique adjoint $\phi^* : \mathcal{B} \to \mathcal{A}$ given by

$$\phi^* b = \bigwedge_{\phi a < b'} a'.$$

Proof: We first show unicity. Let ϕ^* and ϕ^+ be adjoint to ϕ and set $\phi^* b = a$. Then $\phi a' = \phi(\phi^* b)' < b'$ since ϕ and ϕ^* are adjoint. Hence $b < (\phi a')'$. But then, since ϕ^+ is monotone we have that $\phi^+ b < \phi^+(\phi a')' < a = \phi^* b$, where we have used the fact that ϕ and ϕ^+ are adjoint. Interchanging the roles of ϕ^* and ϕ^+ we have that $\phi^* = \phi^+$.

We now show existence. Define $\phi^* b = \bigwedge_{\phi a < b'} a$. We first show that ϕ^* is a hemimorphism. It is trivial that $\phi^* 0 = 0$. Further,

$$\phi^*(\vee_i b_i) = \bigwedge_{\phi a < (\vee_i b_i)'} a = \bigwedge_{\phi a < b_i' \ \forall b_i} a = \vee_i \phi^*(b_i).$$

We now show that ϕ and ϕ^* form an adjoint pair. We have that $\phi^*(\phi a)' = \bigwedge_{\phi x < \phi a} x' < a'$, by considering $x = a$. On the other hand, $\phi(\phi^* b)' = \phi(\bigwedge_{\phi a < b'} a')' = \phi(\bigvee_{\phi a < b'} a) = \bigvee_{\phi a < b'} \phi a < b'$, where we have used the fact that ϕ preserves the supremum, completing the proof.

Example: Let J_a be the canonical injection $J_a : [0, a] \to \mathcal{A}$. We show that $J_a^* J_a = I$ on $[0, a]$ and that $J_a J_a^* = \phi_a$ on $\mathcal{A}$, where ϕ_a is the Sasaki projection defined by $\phi_a x = (x \vee a') \wedge a$. The projections are then self-adjoint and idempotent: $\phi_a = \phi_a^* = \phi_a^2$. Indeed, for $x \in \mathcal{A}$ and $y \in [0, a]$ we have that

$$J_a^* x = \bigwedge_{J_a y < x'} y^r = \Big(\bigvee_{\substack{y<a \\ y<x'}} y \Big)^r = (a \wedge x')^r = (x \vee a') \wedge a$$

and hence $J_a^* J_a y = J_a^* y = y$ (exactly the orthomodularity condition) and

$$J_a J_a^* x = (x \vee a') \wedge a = \phi_a x.$$

Example: We show that a hemimorphism u is an isomorphism if and only if $u^* = u^{-1}$. Indeed, let u be an isomorphism, then $u(u^{-1}b)' = (uu^{-1}b)' = b'$ and $u^{-1}(ua)' = (u^{-1}ua)' = a'$ so that the two conditions on an adjoint pair are satisfied and $u^* = u^{-1}$. Conversely, let $u^{-1} = u^*$, then the first condition imposes that $u^{-1}(ua)' < a'$ which means that $(ua)' < ua'$. Furthermore, setting $b = (ua)'$ for any given a, the second condition imposes that $u(u^{-1}b)' < b'$ which means $(u^{-1}b)' < u^{-1}b'$ and by substitution $(u^{-1}(ua))' < a$ and also $a' < u^{-1}(ua)'$ and so $ua' < (ua)'$. Hence $ua' = (ua)'$ and so u is an isomorphism.

Theorem: The set S of hemimorphisms of a CROC $\mathcal{A}$ into itself, equipped with the composition law and the adjoint defined above, forms a complete Baer *-semigroup.

Proof: (i) It is clear that the composition law of hemimorphisms is associative and that the hemimorphisms $a \mapsto 0$ and $a \mapsto a$ play the role of 0 and I respectively.

(ii) The adjoint operation $\phi \mapsto \phi^*$ is well-defined and $\phi^{**} = \phi$ since the conditions on an adjoint pair are symmetric. Finally $(\psi\phi)^* = \phi^*\psi^*$ since the two conditions are satisfied. Indeed, from $\psi^*(\psi\phi a)' < (\phi a)'$ we derive the first, $\phi^*\psi^*(\psi\phi a)' < \phi^*(\phi a)' < a'$, and we obtain the second, $\psi\phi(\phi^*\psi^* b)' < b'$, by the same kind of reasoning.

(iii) Finally let $M \subset S$ and set $a_i = (\phi_i I)'$ for each $\phi_i \in M$. Define $a = \wedge_i a_i$. We show that the annihilator $\{\psi \mid \psi\phi_i = 0 \quad \forall \phi_i \in M\}$ is identical to the ideal $S\phi_a$, where ϕ_a is the Sasaki projection $\phi_a x = (x \vee a') \wedge a$. Obviously $S\phi_a$ will be contained in the annihilator of M since $\phi_a \phi_i = 0$ for each $\phi_i \in M$. Indeed

$$\phi_a \phi_i x < \phi_a \phi_i I = \phi_a a_i' = (a_i' \vee a') \wedge a = 0.$$

On the other hand, let ψ be in the annihilator of M, $\psi\phi_i = 0$ for all $\phi_i \in M$. We show that $\psi = \psi\phi_a$. Now $\psi(\phi_i I) = 0$ so that $\psi^* I = \psi^*(\psi(\phi_i I))' < (\phi_i I)' = a_i$ and so $\psi^* x < a_i$ for all a_i. Hence $(\phi_a \psi^*)x = \psi^* x$ for all x. This means that $\phi_a \psi^* = \psi^*$, and so $\psi = \psi\phi_a$ by taking the adjoint.

3. The CROC associated to a complete Baer *-semigroup

We can define a partial order relation on the set of projections of a complete Baer *-semigroup by setting $p < q$ if $p = pq$. This order relation can readily be seen to be identical to the set-theoretical inclusion

$$Sp \subset Sq.$$

To each element $f \in S$ we will associate the projection f' which generates the annihilator of f:

$$\{g \mid gf = 0, g \in S\} = Sf'.$$

Such a projection exists by definition and we have the following properties:
(i) If p is a given projection then $p < p''$. Indeed, since in particular $p'p = 0$ then $(p'p)^* = pp' = 0$ and so p is in the annihilator of p'. This means that there exists an f with $p = fp'' = fp''p'' = pp''$.
(ii) If p and q are two projectors such that $p < q$ then $q' < p'$. Indeed by taking the adjoint of $p = pq$ we find that $p = qp$ which implies $q'p = q'(qp) = (q'q)p = 0$ and so $q' < p'$.
From these two properties the map $p \mapsto p''$ is a closure operation and $p' = p'''$. This justifies the following definition: p is a closed projector if $p = p''$. Note that 0 and I are closed since $0' = I$ and $I' = 0$.

Theorem: The set $\mathcal{A}$ of closed projectors of a complete Baer *-semigroup S, equipped with the partial order defined above and the orthogonality map $p \mapsto p'$, is a CROC.

Proof: (i) To show that $\mathcal{A}$ is a complete lattice it suffices to show that there exists a closed projector $\bigwedge_i p_i$, the infimum of any given family $\{p_i\}$, since there is a maximal element I. Now $p < q$ if and only if $Sp \subset Sq$ and so the infimum of a family $\{p_i\}$ must be associated to $\bigcap_i Sp_i$. However, as each p_i is closed this is just the annihilator of the family $\{p_i'\}$ which is by definition generated by some projection p. It therefore remains to show that p is necessarily closed. Since $p < p_i$ we have that $p_i' < p'$ and so $p'' < p_i'' = p_i$ for all p_i. Hence $p'' < p$. On the other hand $p < p''$ as p is a projection

(ii) We now show that the map $p \mapsto p'$ is an orthocomplementation. We have that $p' = p'''$ so that the map is well-defined. The map is trivially involutive and order reversing. Finally $p \wedge p' = 0$ since if $q < p$ and $q < p'$ then $q = qp$ and $q = qp'$ giving $q = qp = qp'p = 0$.

(iii) The orthomodular law states that if $p < q$ then $(q' \vee p) \wedge q = p$. In fact it suffices to show that $(q \wedge p')' \wedge q < p$ since the opposite inequality is trivial; that is we must show that $(q \wedge p')' \wedge q$ is in the annihilator of p'. We use the fact that in general if $pq = qp$ then $pq' = q'p$, $q \wedge p = pq$ and $q \wedge p' = qp'$. Indeed, let $pq = qp$. Then $(q'p)q = (q'q)p = 0$. Hence $q'p = q'pq'$ and so by taking the adjoint $q'p = pq'$. Further, it is simple to show that if $pq = qp$ then pq is a projection, in fact the projection $p \wedge q$ (von Neumann 1950). Now $pqq = pq$ so that $pq < q$ and in the same way $pq < p$. Finally, if $r < p$ and $r < q$ then $r = rp$ and $r = rq$ so that $r = rpq$ and $r < pq$. Now let $p < q$. Then $pq = qp$ so that $q \wedge p' = qp'$. Then $q(q \wedge p') = (q \wedge p')q$ and so $(q \wedge p')' \wedge q = (qp')'q$. But then it is trivial that $(qp')'qp' = 0$, completing the proof.

Theorem: Let $\mathcal{A}$ be a CROC and S the associated Baer *-semigroup. Then the CROC associated to S is exactly $\mathcal{A}$.

Proof: We need to show that the closed projections of S are exactly of the form ϕ_a for some $a \in \mathcal{A}$. We use the fact that $\phi' = \phi_a$ for $a = (\phi I)'$ as shown in section 2 and so $(\phi_a)' = \phi_{a'}$. Each projection of the form ϕ_a is then closed since $(\phi_a)'' = (\phi_{a'})' = \phi_a$. On the other hand, let ϕ be a closed projection: $\phi = \phi''$. Then $\phi' = \phi_a$ for $a = (\phi I)'$ and so

$\phi = \phi'' = (\phi_a)' = \phi_{a'}$, completing the proof.

Note that one cannot pass from a complete Baer *-semigroup to the associated CROC and back again in general. Indeed, let S be any field considered as a complete Baer *-semigoup under multiplication, where we take the identity as the involution. Then there are only two projections, namely 0 and I, since $a^2 = a$ implies $a(a-1) = 0$. Hence all such S have the same associated trivial CROC $\{0, 1\}$. Note that this CROC has only two hemimorphisms as one can send 1 to either 0 or 1.

4. An example

In this final section we will consider the most simple non-trivial CROC which has four elements, namely 0, a, a' and 1. In this case there are sixteen hemimorphisms ϕ, as one can send a and a' independently to an arbitrary element, and set $\phi 1 = \phi a \vee \phi a'$. This example, although very simple, exhibits much of the relevant structure of the set of hemimorphisms. For example, one of the hemimorphisms will be seen to be a projection which is not closed. Further, one can see that the adjoint of a morphism need not be a morphism.

The sixteen hemimorphisms of will be labelled $\phi_{\alpha\beta}$, where α is the image of a' and β is the image of a. Hence, for example, the identity hemimorphism is $\phi_{a'a}$. We give a table giving the adjoint of each hemimorphism and stating whether a given hemimorphism is self-adjoint, idempotent, closed or a morphism.

$\phi_{\alpha\beta}$	$(\phi_{\alpha\beta})^*$	self-adjoint	idempotent	closed	morphism
$0\,0$	$0\,0$	Y	Y	Y	Y
$0\,a$	$0\,a$	Y	Y	Y	Y
$0a'$	$a\,0$	N	N	N	Y
$0\,1$	$a\,a$	N	Y	N	Y
$a\,0$	$0a'$	N	N	N	Y
$a\,a$	$0\,1$	N	Y	N	N
aa'	aa'	Y	N	N	Y
$a\,1$	$a\,1$	Y	N	N	N
$a'0$	$a'0$	Y	Y	Y	Y
$a'a$	$a'a$	Y	Y	Y	Y
$a'a'$	$1\,0$	N	Y	N	N
$a'1$	$1\,a$	N	Y	N	N
$1\,0$	$a'a'$	N	Y	N	Y
$1\,a$	$a'1$	N	Y	N	N
$1a'$	$1a'$	Y	N	N	N
$1\,1$	$1\,1$	Y	Y	N	N

Hence we see that there are four closed projections; ϕ_{00}, ϕ_{0a}, $\phi_{a'0}$ and $\phi_{a'a}$ which regive the original CROC. There is one projection which is not closed, namely ϕ_{11}. Finally there are two morphisms whose adjoints are not morphisms, namely ϕ_{01} and ϕ_{10}.

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